Probability of survival for vicious walkers near a cliff

1989 J. Phys. A: Math. Gen. 22 L609

(http://iopscience.iop.org/0305-4470/22/13/014)

J. Phys. A: Math. Gen. 22 (1989) L609-L613. Printed in the UK

LETTER TO THE EDITOR

Probability of survival for vicious walkers near a cliff

P J Forrester

Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia

Received 12 April 1989

Abstract. The partition function for a model system of N vicious walkers is expressed as a determinant when the walkers are confined to a half-line or a finite interval. For the walkers on the half-line, the probability of survival, the probability of a reunion and the conditional probability of a reunion are obtained. The two-point correlation near and parallel to a boundary of fixed like spins in the magnetised phase of the two-dimensional Ising model is conjectured to be proportional to the probability of a reunion near the boundary for $N=2$ and thus decay as $e^{-r/\xi}/r^5$ for large distances r.

The lock step model of vicious walkers on a one-dimensional lattice allows each walker at the tick of a clock to move either one lattice site to the left or one lattice site to the right. The only restriction is that no two walkers may arrive at the same lattice site or pass one another. It was shown by Fisher (1984) that the partition function for this model in free boundary conditions can be expressed as a determinant. This result was used by Fisher (1984) to provide physically instructive heuristic arguments predicting quantitative features of the critical properties of wetting in two dimensions and related phenomena.

Subsequently, it was proved by this author (Forrester 1989) that the partition function with the vicious walkers in periodic boundary conditions could also be expressed as a determinant (provided that N, the number of walkers, is odd). This result allows simple formulae for the partition function and correlation functions of a model of the incommensurate-commensurate phase transition to be provided in a finite system. The primary purpose of this letter is to prove that the partition function for a further two different types of boundary conditions can again be expressed as determinants. These further boundary conditions are:

(i) a hardwall boundary condition on one end (which can be thought of as a cliff over which the walkers can fall to their death) and a free boundary condition on the other;

(ii) two hardwall boundary conditions.

In situation (i) the system is a semi-infinite one-dimensional regular lattice (with lattice sites at the positive integers, say), while in (ii) it is a finite lattice (with lattice sites at the integers from 1 to M, say). Initially the walkers are at the lattice sites

$$l_1' < l_2' < \ldots < l_N' \tag{1}$$

which are required to be of the same parity (i.e. all even or all odd). After n time intervals the walkers are to arrive at the lattice sites

$$l_1 < l_2 < \ldots < l_N. \tag{2}$$

The walkers are subject to the constraints that they must move either to the left or to the right at each tick of the clock without arriving at the same lattice site as any other walker. (Having required that the initial sites (1) all have the same parity, this implies that paths cannot cross.) Furthermore, each walker must remain within the domain specified by (i) and (ii) above.

The partition function for these models can be written

$$_{\mathrm{BC}}Z(l_1',\ldots,l_N'\,|\,l_1,\ldots,l_N\,;\,n)=\sum_{P\in\wp}\prod_{j=1}^{N}W_n(l_j'\,|\,l_j). \tag{3}$$

Here $\wp$ denotes the set of all allowed paths from l_j' to l_j for each $j=1,\ldots,N$ and BC denotes the particular boundary condition: wall-free (WF) or wall-wall (WW). Also

$$W_n(l_j'\,|\,l_j)=\omega_{-1}^{l}\omega_1^{r}$$

where l is the number of steps to the left and r the number of steps to the right in the jth path.

The following result extends Fisher's master formula (Fisher 1984, equation (5.1)).

Theorem

$$_{\mathrm{BC}}Z(l_1',\ldots,l_N'\,|\,l_1,\ldots,l_N\,;\,n)=\det[_{\mathrm{BC}}Q_n(l_j'\,|\,l_k)]_{j,k=1,\ldots,N} \tag{4}$$

where

$$_{\mathrm{WF}}Q_n(l_j'\,|\,l_k)=\int_0^1 \mathrm{e}^{-2\pi\mathrm{i}(l_j'-l_k)\theta}[\phi(2\pi\theta)]^n\,\mathrm{d}\theta-\int_0^1 \mathrm{e}^{-2\pi\mathrm{i}(l_j'+l_k)\theta}[\phi(2\pi\theta)]^n\,\mathrm{d}\theta \tag{5}$$

and

$$\begin{aligned}_{\mathrm{WW}}Q_n(l_j'\,|\,l_k)=&\sum_{a=1}^{M}\mathrm{e}^{-2\pi\mathrm{i}(l_j'-l_k)a/2M}[\phi(2\pi a/M)]^n\\&-\sum_{a=1}^{M}\mathrm{e}^{-2\pi\mathrm{i}(l_j'+l_k)a/2M}[\phi(2\pi a/M)]^n.\end{aligned} \tag{6}$$

In (5) and (6)

$$\phi(\theta)=\omega_{-1}\,\mathrm{e}^{-\mathrm{i}\theta}+\omega_1\,\mathrm{e}^{\mathrm{i}\theta}. \tag{7}$$

Remark. $_{\mathrm{BC}}Q_n(l_j'\,|\,l_k)$ is the partition function for a single walker going from l_j' to l_k in n steps with the specified boundary condition.

Proof. The key to the proof is to note that the partition function in each case is uniquely defined as the solution of a multivariable difference equation. If the *notation*

$$_{\mathrm{BC}}Z(l_1',\ldots,l_N'\,|\,l_1,\ldots,l_N\,;\,n):={_{\mathrm{BC}}F(l_1\,;\,n)}\,{_{\mathrm{BC}}F(l_2\,;\,n)}\ldots{_{\mathrm{BC}}F(l_N\,;\,n)} \tag{8}$$

is introduced, then by considering the possible ways of constructing the partition function with n steps from that with $n-1$ steps, we see

$$\begin{aligned}&_{\mathrm{BC}}Z(l_1',\ldots,l_N'\,|\,l_1,\ldots,l_N\,;\,n)\\&\quad=\prod_{j=1}^{N}(\omega_{-1}\,{_{\mathrm{BC}}F(l_j-1;\,n-1)}+\omega_{-1}\,{_{\mathrm{BC}}F(l_j+1;\,n-1)}).\end{aligned} \tag{9}$$

The right-hand side of (9) only has meaning when the product is expanded and the definition (8) used.

The difference equation (9) is to be solved subject to the initial condition

$$ {}_{\mathrm{BC}}Z(l_1', \ldots, l_N' \mid l_1, \ldots, l_N; 0) = \prod_{j=1}^{N} \delta_{l_j', l_j} \tag{10} $$

the non-intersecting condition

$$ {}_{\mathrm{BC}}Z(l_1', \ldots, l_N' \mid l_1, \ldots, l_j, \ldots, l_k, \ldots, l_N; n) = 0 \qquad \text{if } l_j = l_k \tag{11} $$

and the boundary conditions

$$ {}_{\mathrm{WF}}Z(l_1', \ldots, l_N' \mid l_1, \ldots, l_N; n) = 0 \qquad \text{if } l_k = 0, k = 1, \ldots, N \tag{12} $$

$$ {}_{\mathrm{WW}}Z(l_1', \ldots, l_N' \mid l_1, \ldots, l_N; n) = 0 \qquad \text{if } l_k = 0 \text{ or } M, k = 1, \ldots, N. \tag{13} $$

The equations (9)-(13) uniquely determine ${}_{\mathrm{WF}}Z$ and ${}_{\mathrm{WW}}Z$.

The reader is referred to Forrester (1989) for the (straightforward) details of proving that (4) satisfies (9). The initial condition (10) follows from the initial condition for the single walker partition function

$$ {}_{\mathrm{BC}}Q_0(l_j' \mid l_k) = \delta_{l_j', l_k} \tag{14} $$

and the boundary conditions (12) and (13) follow similarly from

$$ {}_{\mathrm{WF}}Q_n(l_j' \mid 0) = {}_{\mathrm{WW}}Q_n(l_j' \mid 0) = {}_{\mathrm{WW}}Q_n(l_j' \mid M) = 0. \tag{15} $$

The non-intersecting condition (11) is a consequence of the fact that a determinant vanishes whenever two rows are equal. The theorem is thus established. □

As detailed in Forrester (1989), the continuum limit can be taken to obtain a model of vicious Brownian walkers in the particular boundary conditions. We find

$$ \begin{aligned} {}_{\mathrm{WF}}Q_n(l_j' \mid l_k) &\mapsto {}_{\mathrm{WF}}Q_t(x_j' \mid x_k) \\ &:= \frac{\mathrm{e}^{-\sigma t}}{(2\pi D t)^{1/2}} \left(\mathrm{e}^{(-x_j' - x_k)^2/2Dt} - \mathrm{e}^{-(x_j' + x_k)^2/2Dt}\right) \end{aligned} \tag{16} $$

$$ \begin{aligned} {}_{\mathrm{WW}}Q_n(l_j' \mid l_k) &\mapsto {}_{\mathrm{WW}}Q_t(x_j' \mid x_k) \\ &:= \frac{\mathrm{e}^{-\sigma t}}{2L} [\theta_3(\pi(x_j' - x_k)/2L; \mathrm{e}^{-\pi^2 Dt/2L^2}) \\ &\quad - \theta_3(\pi(x_j' + x_k)/2L; \mathrm{e}^{-\pi^2 Dt/2L^2})] \end{aligned} \tag{17} $$

where D denotes the diffusion constant, t the time and (in (17)) L the length of the system. These formulae, together with (4)-(6), finalise our extensions of Fisher's (1984) master equation.

Let us now explore some of the consequences of (4) and further consider the case BC = WF in the continuum limit. With

$$ x_j' = aj \tag{18} $$

so that the walkers are initially equally spaced, from (4) and (16) we see

$$ \begin{aligned} &{}_{\mathrm{WF}}Z(a, 2a, \ldots, Na \mid l_1, \ldots, l_N; n) \\ &\quad = \left(\frac{2}{\pi D t}\right)^{N/2} \exp\left\{-\frac{1}{2dt}\left(\sum_{j=1}^{N} (x_j)^2 + (aj)^2\right) - \sigma N t\right\} \\ &\quad \times \det[\sinh jax_k / Dt]_{j,k} = 1, \ldots, N. \end{aligned} \tag{19} $$

The large Dt expansion of the determinant can be obtained as follows. In the jth row replace $\sinh jax_k/Dt$ by its Taylor polynomial approximation

$$\sum_{n=1}^{j} \frac{1}{(2n-1)!}(jax_k/Dt)^{2n-1}. \tag{20}$$

Subtracting a suitable factor of the first row from each of the other rows gives for each row $j \geq 2$ the entries

$$\sum_{n=2}^{j} \frac{1}{(2n-1)!}(jax_k/Dt)^{2n-1}. \tag{21}$$

Now subtracting the new row 2 from each of the rows $j = 3, \ldots, N$ gives these latter rows the entries

$$\sum_{n=3}^{j} \frac{1}{(2n-1)!}(jax_k/Dt)^{2n-1}. \tag{22}$$

Continuing in this fashion we see that for large Dt

$$\begin{aligned} &\det[\sinh jax_k/Dt]_{j,k=1,\ldots,N} \\ &\quad \sim A_N\left(\frac{1}{Dt}\right)^{N^2} \det[x_k^{2j-1}]_{j,k=1,\ldots,N} \\ &\quad = A_N\left(\frac{1}{dT}\right)^{N^2}\left(\prod_{k=1}^{N} x_k\right)\prod_{1\leq j<k\leq N}((x_k)^2-(x_j)^2) \end{aligned} \tag{23}$$

where

$$A_N = a^N N!\left[\prod_{j=1}^{N} 1/(2j-1)!\right] \tag{24}$$

and use has been made of the van der Monde determinant expansion.

The results (19) and (23) allow the large Dt behaviour of the probability of a reunion to be calculated. A reunion is the term used for the event in which the spacing of the walkers in their final configuration is the same as in their initial configuration. Thus

$$x_k = \mu + ka \qquad k = 1, 2, \ldots, N \tag{25}$$

where $\mu \geq 0$ specifies the displacement of the final configuration from the wall. Denoting the probability by $r_t^{(N)}(\mu)$, we see from (19) and (23) that the leading-order behaviour for large Dt and large μ is

$$r_t^{(N)}(\mu) \sim B_N\, \mathrm{e}^{-\sigma NT}\left(\frac{1}{Dt}\right)^{N^2+N/2} \mathrm{e}^{-N\mu^2/2Dt}\mu^{N(N+1)/2}. \tag{26}$$

The constant B_N can be specified if required. The exponent $N^2 + N/2$ with $N = 2$ agrees with a result of Fisher (1984, equation (5.10)).

As an application of this result, consider the two-dimensional Ising model in a half-plane with the boundary spins all fixed in the one direction. Extending the arguments of Fisher (1984, § 11) the two-point correlation parallel to the boundary in the magnetised phase, $C(r)$ say, should be proportional to $r_t^{(2)}(\mu)$ (with $t = r$). Thus from (26)

$$C(r) = A(T, d)\, \mathrm{e}^{-r/\xi(T,d)}/r^5 \tag{27}$$

where T is the reduced temperature and d is the number of rows from the boundary, A is the amplitude and ξ the correlation length. The exponent 5 is to be contrasted to the corresponding exponent 2 in the bulk (see e.g. Fisher 1984, equation (11.3)).

By integrating (26) over μ from 0 to ∞ the large Dt behaviour of the probability $R_t^{(N)}$ of a reunion anywhere is specified. Thus

$$R_t^{(N)} \sim C_N \, \mathrm{e}^{-\sigma NT} \left(\frac{1}{Dt}\right)^{(3N^2+N-2)/4}. \tag{28}$$

The probability of survival irrespective of the particular final configuration $P_t^{(N)}$ is calculated for large Dt from (19) and (23) by integrating over the region

$$0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N < \infty. \tag{29}$$

However, the integrand is positive in this region and its absolute value is symmetric in each x_k. Thus, replacing the integrand by its absolute value, the region of integration (29) can be replaced by integrating each x_k over the half-line $0 \leqslant x_k < \infty$ provided we divide by $N!$. Changing variables

$$x_j = \sqrt{Dt}\, X_j \tag{30}$$

then gives

$$P_t^{(N)} \sim D_N \, \mathrm{e}^{-\alpha Nt} \left(\frac{1}{Dt}\right)^{N^2/2} \tag{31}$$

where the constant D_N can easily be specified if required. The conditional probability of survival

$$S_t^{(N)} := R_t^{(N)} / P_t^{(N)} \tag{32}$$

for large Dt, from (28) and (31) therefore behaves as

$$S_t^{(N)} \sim E_N \left(\frac{1}{Dt}\right)^{N^2+N-2)/4}. \tag{33}$$

The exponent $(N^2+N-2)/4$ is precisely that calculated by Fisher (1984, equation (4.5)) for $S_t^{(N)}$ in free boundary conditions. The wall therefore has no effect on the leading behaviour of this quantity. However, the exponents in (26), (28) and (31) are different from the corresponding exponents found by Fisher (1984) in free boundary conditions.

References

Fisher M E 1984 *J. Stat. Phys.* **34** 669
Forrester P J 1989 *J. Phys. A: Math. Gen.* **22** in press